

Vacuum Polarization in QED with World-Line Methods

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Abstract

Motivated by several recent papers on string-inspired calculations in QED, we here present our own use of world-line techniques in order to calculate the vacuum polarization and effective action in scalar and spinor QED with external arbitrary constant electromagnetic field configuration.

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I. INTRODUCTION

String-inspired methods in QFT were initiated by Bern and Kosower [1], who applied them to compute one-loop amplitudes in various field theories. These authors and Strassler [2] then recognized that some of the well-known vacuum processes in QED and QCD can be computed rather easily with the aid of one-dimensional path integrals for relativistic point particles. Similar techniques and results are also contained in the monograph by Polyakov [3]. String-inspired methods, particularly in QED, were then extensively studied in a series of papers by Schmidt, Schubert and Reuter, cf., e.g., [4], where the state of the art is reviewed extensively. There are also contributions by McKeon [5] and various co-authors who have proved that world-line methods are extremely useful.

In the present article we will try to compactify their work and present some new representations which will finally permit us to write the one-loop diagram tied to an arbitrary number of off-shell photons - together with an applied external constant electromagnetic field of any configuration - in a so far unknown manner. This highly condensed form for the one-loop vacuum process turns out to be directly applicable to various limiting situations, e.g., turning off the off-shell photon lines will bring us back to the effective action of QED which in the low-frequency limit yields the Heisenberg-Euler Lagrangian. Another process should be photon splitting in a prescribed constant magnetic field as calculated by Adler and Schubert [6].

II. VACUUM POLARIZATION WITHOUT EXTERNAL FIELDS - SCALAR QED.

Our starting point is the one loop action, which we write in a path integral representation,

$$\Gamma[A] = -i \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x e^{i \int_0^T d\tau \left[-\frac{\dot{x}^2}{4} - eA\dot{x} - m^2 \right]}, \quad (1)$$

where \mathcal{N} is determined by

$$\mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x e^{-i \int_0^T d\tau \frac{\dot{x}^2}{4}} = -\frac{i}{(4\pi T)^2} V_4. \quad (2)$$

In (1), A denotes a superposition of external plus radiation field. If we then expand the radiation field in plane waves, $A_\mu(x) = \sum_{i=1}^N \varepsilon_i^\mu e^{ik_i x}$, we obtain

$$\begin{aligned} \Gamma_N[k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] &= -i (-ie)^N \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x e^{i \int_0^T d\tau \left[-\frac{\dot{x}^2}{4} - eA\dot{x} - m^2 \right]} \\ &\quad \cdot \prod_{i=1}^N \int_0^T dt_i \varepsilon_i \cdot \dot{x}(t_i) e^{ik_i \cdot x(t_i)} \end{aligned} \quad (3)$$

Hence, without external field and keeping only terms linear in ε_i , (3) becomes

$$\Gamma_N = -ie^N \int_0^\infty \frac{dT}{T} \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x e^{i \int_0^T d\tau \left[-\frac{\dot{x}^2}{4} - m^2 \right]} \left(\prod_{i=1}^N \int_0^T dt_i \right) \cdot$$

$$\cdot e^{i \sum_{j=1}^N (k_j \cdot x(t_j) - i\varepsilon_j \cdot \dot{x}(t_j))} \Big|_{\text{linear in each } \varepsilon_j} \quad (4)$$

Introducing the source for x^μ

$$J^\mu(\tau) = i \sum_{j=1}^N \left(k_j^\mu - \varepsilon_j^\mu \frac{\partial}{\partial t_j} \right) \delta(\tau - t_j), \quad (5)$$

we can rewrite (4) as

$$\Gamma_N = -ie^N \int_0^\infty \frac{dT}{T} e^{-im^2 T} \left(\prod_{i=1}^N \int_0^T dt_i \right) \mathcal{N} \int_{x(T)=x(0)} \mathcal{D}x e^{\frac{i}{4} \int_0^T d\tau x \frac{d^2}{d\tau^2} x} e^{\int_0^T J(\tau) x(\tau)}. \quad (6)$$

The operator $d^2/d\tau^2$, acting on $x(\tau)$ with periodical boundary condition $x(T) = x(0)$, has zero modes x_0 . These modes will be separated from their orthogonal non-zero mode partners by writing $x(\tau) = x_0 + \xi(\tau)$ with $\int_0^T d\tau \xi^\mu(\tau) = 0$, i.e., $\int_0^T d\tau x^\mu(\tau) = x_0^\mu$ and $\int \mathcal{D}x = \int d^4 x_0 \int \mathcal{D}\xi$.

Since $x(\tau)$ (and therefore $\xi(\tau)$) is periodic we can write

$$x(\tau) = x_0 + \sum_{n \neq 0} c_n e^{\frac{2\pi i n}{T} \tau}. \quad (7)$$

Now with the use of (2), equation (6) may be written as

$$\Gamma_N = -(2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T^3} e^{-im^2 T} \left(\prod_{i=1}^N \int_0^T dt_i \right) \frac{\int \mathcal{D}\xi e^{\frac{i}{4} \int_0^T d\tau \xi \frac{d^2}{d\tau^2} \xi} e^{\int_0^T J \cdot \xi}}{\int \mathcal{D}\xi e^{\frac{i}{4} \int_0^T d\tau \xi \frac{d^2}{d\tau^2} \xi}}$$

$$= -(2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T^3} e^{-im^2 T} \prod_{i=1}^N \int_0^T dt_i \cdot$$

$$\cdot e^{\frac{i}{2} \int_0^T d\tau \int_0^T d\tau' J^\mu(\tau) G_{\mu\nu}(\tau, \tau') J^\nu(\tau')}, \quad (8)$$

where the Green's function $G_{\mu\nu}$ and its properties are given by

$$G_{\mu\nu}(\tau, \tau') = g_{\mu\nu} G(\tau, \tau'), \quad \frac{1}{2} \partial_\tau^2 G(\tau, \tau') = \delta(\tau - \tau') - \frac{1}{T}$$

$$G(\tau, \tau') = |\tau - \tau'| - \frac{(\tau - \tau')^2}{T} + \text{const}; \quad G(\tau, \tau') = G(\tau - \tau') = G(\tau', \tau)$$

$$\partial_\tau G(\tau, \tau') \equiv \dot{G}(\tau, \tau') = \text{sign}(\tau - \tau') - \frac{2(\tau - \tau')}{T}; \quad \dot{G}(\tau, \tau') = -\dot{G}(\tau', \tau). \quad (9)$$

Note the generic structure expressed in (8), where particle and off-shell photons are factorized in such a way that the free (i.e., sans external field) scalar particle circulating in the loop becomes multiplied by the exponential term which is solely due to the photons tied to the loop. With $J^\mu(\tau)$ given in (5) we finally obtain

$$\Gamma_N[k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] = -(2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T^3} e^{-im^2 T} \prod_{i=1}^N \int_0^T dt_i \quad (10)$$

$$\exp -\frac{i}{2} \sum_{i,j=1}^N \left[k_i \cdot k_j G(t_i, t_j) - k_i \cdot \varepsilon_j \frac{\partial}{\partial t_j} G(t_i, t_j) - k_j \cdot \varepsilon_i \frac{\partial}{\partial t_i} G(t_i, t_j) + \varepsilon_i \cdot \varepsilon_j \frac{\partial^2}{\partial t_i \partial t_j} G(t_i, t_j) \right].$$

Since $G(\tau, \tau) = 0$, $\dot{G}(\tau, \tau)$, there are no terms with k_i^2 and $\varepsilon_i \cdot k_i$, i.e., without use of on-shell conditions.

As an example we just note that for $N=2$ we obtain, after keeping only terms linear in ε_i ,

$$\Gamma_2[k_1, \varepsilon_1; k_2, \varepsilon_2] = -(2\pi)^4 \delta^4(k_1 + k_2) e^2 \int_0^\infty \frac{dT}{(4\pi T)^2} e^{-im^2 T} \int_0^T dt_1 e^{ik_1^2 G(t_1)} \{ (k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2) \} \dot{G}^2(t_1). \quad (11)$$

This expression is manifestly gauge invariant. Upon using $G(t_1) = -\frac{t_1^2}{T} + t_1$, $\dot{G}(t_1) = -\frac{2t_1}{T} + 1$ and substituting $v = \frac{2t_1}{T} - 1$, we obtain

$$\Gamma_2[k_1, \varepsilon_1; k_2, \varepsilon_2] = -(2\pi)^4 \delta^4(k_1 + k_2) e^2 [(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2)] \cdot \int_0^\infty \frac{dT}{2(4\pi)^2 T} e^{-im^2 T} \int_{-1}^1 dv v^2 e^{ik_1^2 \frac{T}{4}(1-v^2)}. \quad (12)$$

At this point we can make contact with the vacuum polarization digram in scalar QED:

$$\Gamma_2 = (2\pi)^4 \delta^4(k_1 + k_2) \varepsilon_\mu \Pi^{\mu\nu} \varepsilon_\nu, \quad (13)$$

where, after renormalization, we obtain for $\Pi_{\mu\nu} = (g_{\mu\nu} k^2 - k_\mu k_\nu) \Pi(k^2)$

$$\Pi(k^2) = \frac{\alpha}{4\pi} \int_0^1 dx (2x-1)^2 \ln \left[1 - \frac{k^2}{m^2} x(1-x) \right]. \quad (14)$$

All these results have been reproduced here without the use of operator field theory.

III. VACUUM POLARIZATION IN SCALAR QED WITH EXTERNAL FIELDS.

Following the previous discussion on the free-field case we will briefly describe the effect of an external electromagnetic field on the single-loop process in scalar QED. Our starting point is a short review of the results achieved earlier by one of the present authors [7]. There it was shown that the action in presence of a constant electromagnetic background field is given by

$$\Gamma_N [k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] = - (2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T^3} e^{-im^2 T} \cdot \prod_{i=1}^N \int_0^T dt_i \frac{det_P'^{1/2} \left[\frac{d^2}{d\tau^2} \right]}{det_P'^{1/2} \left[\frac{d^2}{d\tau^2} - 2eF \frac{d}{d\tau} \right]} e^{\frac{i}{2} \int_0^T d\tau \int_0^T d\tau' J^\mu(\tau) G_{\mu\nu}(\tau, \tau') J^\nu(\tau')} \quad (15)$$

where the Green's function equation is now slightly modified:

$$\frac{1}{2} \ddot{G} - eF \dot{G} = \delta(\tau - \tau') - \frac{1}{T}. \quad (16)$$

det'_P stands for non-zero modes in the eigenvalue problem with periodic boundary conditions, just as in the previous chapter. Also note that our former expression (8) has maintained its structure, the only difference being that the former freely circulating particle is now propagating in the external field expressed by the field-dependant determinant in (15). As shown in [4] we can write for the ratio of the two determinants in (15)

$$\frac{det_P'^{1/2} \left[\frac{d^2}{d\tau^2} \right]}{det_P'^{1/2} \left[\frac{d^2}{d\tau^2} - 2eF \frac{d}{d\tau} \right]} = \frac{e^2 ab T^2}{\sin(ebT) \sinh(eaT)} \quad (17)$$

so that the free action of equation (8) is modified according to

$$\Gamma_N [k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] = - (2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T} \frac{e^2 ab e^{-im^2 T}}{\sin(ebT) \sinh(eaT)} \cdot \prod_{i=1}^N \int_0^T dt_i e^{\frac{i}{2} \int_0^T d\tau \int_0^T d\tau' J^\mu(\tau) G_{\mu\nu}(\tau, \tau') J^\nu(\tau')} \quad (18)$$

with $G_{\mu\nu}(\tau, \tau')$ given as in [7], and a and b are expressed by the invariants

$$a^2 = \left(\mathcal{F}^2 + \mathcal{J}^2 \right)^{\frac{1}{2}} + \mathcal{F}, b^2 = \left(\mathcal{F}^2 + \mathcal{J}^2 \right)^{\frac{1}{2}} - \mathcal{F}; \mathcal{F} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \mathcal{J} = -\frac{1}{4} F_{\mu\nu}^* F^{\mu\nu} \quad (19)$$

Having recognized similar structures in the free-field case we now replace (10) by

$$\Gamma_N [k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] = - (2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{(4\pi)^2 T} \frac{e^2 ab e^{-im^2 T}}{\sin(ebT) \sinh(eaT)} \prod_{i=1}^N \int_0^T dt_i \cdot \exp -\frac{i}{2} \sum_{i,j=1}^N \left[k_i G(t_i, t_j) k_j - k_i \frac{\partial}{\partial t_j} G(t_i, t_j) \varepsilon_j - \varepsilon_i \frac{\partial}{\partial t_i} G(t_i, t_j) k_j + \varepsilon_i \frac{\partial^2}{\partial t_i \partial t_j} G(t_i, t_j) \varepsilon_j \right]. \quad (20)$$

Our next task is to find Γ_2 . However, since the procedure to arrive at a manifestly gauge invariant expression is straightforward, we just report our findings:

$$\Gamma_2 [k_1, \varepsilon_1; k_2, \varepsilon_2] = (2\pi)^4 \delta^4 (k_1 + k_2) e^2 \int_0^\infty \frac{dT}{32\pi^2} \int_{-1}^1 dv \frac{e^2 ab e^{-im^2 T}}{\sin(ebT) \sinh(eaT)} e^{i\Psi} \cdot [(\varepsilon_1 \rho k) (\varepsilon_2 \rho k) - (\varepsilon_1 \lambda k) (\varepsilon_2 \lambda k) - (\varepsilon_1 \rho \varepsilon_2) (k \rho k)], \quad (21)$$

where $k_1 \equiv k \equiv -k_2$ and e.g., $(\varepsilon_1 \rho k) \equiv \varepsilon_1^\mu \rho_{\mu\nu} k^\nu$, etc. The explicit expressions for ρ and λ are given by

$$\rho = C^2 \zeta_3 - B^2 \zeta_4, \quad \lambda = C \zeta_1 - B \zeta_2, \quad (22)$$

$$\begin{aligned} \text{where } \zeta_1 &= \frac{\cosh(eaT) - \cosh(eaTv)}{\sinh(eaT)}, & \zeta_2 &= \frac{\cos(ebT) - \cos(ebTv)}{\sin(ebT)}, \\ \zeta_3 &= \frac{\sinh(eaTv)}{\sinh(eaT)}, & \zeta_4 &= \frac{\sin(ebTv)}{\sin(ebT)}. \end{aligned}$$

The matrices $C_{\mu\nu}$ and $B_{\mu\nu}$ are defined in [8]:

$$F_{\mu\nu} = C_{\mu\nu}a + B_{\mu\nu}b, F_{\mu\nu}^* = C_{\mu\nu}b - B_{\mu\nu}a \quad (23)$$

and can be expressed in terms of the field strength tensors $F_{\mu\nu}, F_{\mu\nu}^*$ and the invariants a and b . In the special situation of parallel electromagnetic fields E, H along the third direction, $C_{\mu\nu}$ and $B_{\mu\nu}$ are simply given by $C_{\mu\nu} = g_\mu^0 g_\nu^3 - g_\nu^0 g_\mu^3$, $B_{\mu\nu} = g_\mu^2 g_\nu^1 - g_\nu^2 g_\mu^1$. However, nowhere in deriving our expression did we employ this special field configuration. Therefore, our result is quite general and was already considered in [8]. The same will be true for spinor QED, which will be our central result presented in the final chapter. Finally we have to define the phase in (21):

$$\Psi = \frac{1}{2} \left[\frac{(kB^2k)}{eb} \frac{\cos(ebT) - \cos(ebTv)}{\sin(ebT)} + \frac{(kC^2k)}{ea} \frac{\cosh(eaT) - \cosh(eaTv)}{\sinh(eaT)} \right]. \quad (24)$$

IV. VACUUM POLARIZATION IN SPINOR QED WITH EXTERNAL FIELDS.

Here we will start right away with the one-loop action

$$\Gamma[A] = \frac{1}{8\pi^2} \int_0^\infty \frac{dT}{T^3} \frac{\int \mathcal{D}x e^{i \int_0^T d\tau \left[-\frac{\dot{x}^2}{4} - eA\dot{x} - m^2 \right]}}{\int \mathcal{D}x e^{-i \int_0^T d\tau \frac{\dot{x}^2}{4}}} \frac{\int \mathcal{D}\psi e^{\int_0^T d\tau \left[-\frac{1}{2} \psi_\mu \dot{\psi}^\mu + eF_{\mu\nu} \psi^\mu \psi^\nu \right]}}{\int \mathcal{D}\psi e^{-\frac{1}{2} \int_0^T d\tau \psi \dot{\psi}}}, \quad (25)$$

where we have introduced the four Grassmann variables $\psi_\mu(\tau)$ which anticommute.

When we use steps similar to those which took us to equation (18), we arrive again at the background-field one-loop function multiplied by the off-shell photons tied to the spinor particle loop:

$$\begin{aligned} \Gamma_N[k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] &= (2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{8\pi^2 T^3} e^{-im^2 T} \\ &\cdot \prod_{i=1}^N \int_0^T dt_i \int d\theta_i d\bar{\theta}_i \frac{\det^{1/2} \left[\frac{d^2}{d\tau^2} \right]}{\det^{1/2} \left[\frac{d^2}{d\tau^2} - 2eF \frac{d}{d\tau} \right]} \frac{\det^{1/2} \left[\frac{1}{2} \frac{d}{d\tau} - eF \right]}{\det^{1/2} \left[\frac{1}{2} \frac{d}{d\tau} \right]}. \end{aligned} \quad (26)$$

$$\cdot \exp \left\{ \frac{i}{2} \int_0^T d\tau \int_0^T d\tau' J^\mu(\tau) G_{\mu\nu}(\tau, \tau') J^\nu(\tau') - \frac{1}{4} \int_0^T d\tau \int_0^T d\tau' \eta^\mu(\tau) \bar{G}_{F\mu\nu}(\tau, \tau') \eta^\nu(\tau') \right\},$$

with

$$\left(\frac{1}{2} \frac{d}{d\tau} - eF \right) \bar{G}_F(\tau, \tau') = \delta(\tau - \tau')$$

$$\text{and } \eta^\mu(\tau) = \sqrt{2} \sum_{j=1}^N (\theta_j \varepsilon_j^\mu - i \bar{\theta}_j k_j^\mu) \delta(\tau - t_j). \quad (27)$$

As usual, θ_i and $\bar{\theta}_i$ denote independent anticommuting Grassmann variables.

Calculating the determinants yields for the one-loop action with arbitrary constant electromagnetic field

$$\Gamma_N[k_1, \varepsilon_1, \dots, k_N, \varepsilon_N] = (2\pi)^4 \delta^4 \left(\sum_{i=1}^N k_i \right) e^N \int_0^\infty \frac{dT}{8\pi^2 T} e^{-im^2 T} \cdot \prod_{i=1}^N \int_0^T dt_i \int d\theta_i d\bar{\theta}_i e^2 ab \frac{\cos(ebT) \cosh(eaT)}{\sin(ebT) \sinh(eaT)} \cdot \exp \left\{ \frac{i}{2} \int_0^T d\tau \int_0^T d\tau' J^\mu(\tau) G_{\mu\nu}(\tau, \tau') J^\nu(\tau') - \frac{1}{4} \int_0^T d\tau \int_0^T d\tau' \eta^\mu(\tau) \bar{G}_{F\mu\nu}(\tau, \tau') \eta^\nu(\tau') \right\}. \quad (28)$$

Expressing the sources in the exponential according to equation (5) and (27) we obtain

$$\exp \left\{ -\frac{i}{2} \sum_{i,j=1}^N [k_i G(t_i, t_j) k_j - k_i \frac{\partial}{\partial t_j} G(t_i, t_j) \varepsilon_j \bar{\theta}_j \theta_j - \bar{\theta}_i \theta_i \varepsilon_i \frac{\partial}{\partial t_i} G(t_i, t_j) k_j + \right. \\ \left. + \bar{\theta}_i \theta_i \bar{\theta}_j \theta_j \varepsilon_i \frac{\partial^2}{\partial t_i \partial t_j} G(t_i, t_j) \varepsilon_j] - \frac{1}{2} \sum_{i,j=1}^N [\theta_i \theta_j \varepsilon_i \bar{G}_F(t_i, t_j) \varepsilon_j - \right. \\ \left. - i \bar{\theta}_i \bar{\theta}_j \varepsilon_i \bar{G}_F(t_i, t_j) k_j - i \bar{\theta}_i \theta_j k_i \bar{G}_F(t_i, t_j) \varepsilon_j - \bar{\theta}_i \bar{\theta}_j k_i \bar{G}_F(t_i, t_j) k_j] \right\}. \quad (29)$$

Formula (28) together with (29) is our most general representation for the spin- $\frac{1}{2}$ action with external fields. Let us now turn to the special case $N = 2$. Here we obtained as a check for the free-field case

$$\Gamma_2[k_1, \varepsilon_1; k_2, \varepsilon_2] = (2\pi)^4 \delta^4(k_1 + k_2) e^2 \int_0^\infty \frac{dT}{8\pi^2 T^2} e^{-im^2 T} \int_0^T dt_1 e^{ik_1^2 T} G(t_1) \cdot \left\{ [(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2)] (\dot{G}^2 - 1) \right\} = \quad (30)$$

$$= - (2\pi)^4 \delta^4(k_1 + k_2) e^2 [(k_1 \cdot \varepsilon_2)(k_2 \cdot \varepsilon_1) - (\varepsilon_1 \cdot \varepsilon_2)(k_1 \cdot k_2)] \cdot \int_0^\infty \frac{dT}{16\pi^2 T} e^{-im^2 T} \int_{-1}^1 dv (1 - v^2) e^{ik_1^2 \frac{T}{4} (1 - v^2)}. \quad (31)$$

Written in the form $\Gamma_2 = (2\pi)^4 \delta^4(k_1 + k_2) \varepsilon_\mu \Pi^{\mu\nu} \varepsilon_\nu$, this yields the well-known result for the spin- $\frac{1}{2}$ polarization tensor

$$\Pi^{\mu\nu} = (k^2 g^{\mu\nu} - k^\mu k^\nu) \Pi(k^2),$$

where $\Pi(k^2) = \frac{2\alpha}{\pi} \int_0^1 dx x (1-x) \ln \left[1 - \frac{k^2}{m^2} x (1-x) \right].$ (32)

After relatively straightforward calculations which are much easier and shorter than anything published in the literature we finally end up with the following form for the spin- $\frac{1}{2}$ polarization tensor with a prescribed constant electromagnetic field of any configuration:

$$\begin{aligned} \Pi^{\mu\nu}(k) = & \frac{e^2}{16\pi^2} \int_0^\infty dT T e^{-im^2 T} \frac{e^2 ab}{\sin(ebT) \sinh(eaT)} \int_{-1}^{+1} dv e^{i\Psi} \cdot \\ & \cdot \{ \mathcal{F}_1 (g^{\mu\nu} k^2 - k^\mu k^\nu) + \mathcal{F}_2 [(Ck)^\mu (Bk)^\nu + (Ck)^\nu (Bk)^\mu] + \\ & + \mathcal{F}_3 [(C^2 k)^\mu (C^2 k)^\nu - (C^2)^{\mu\nu} (k C^2 k)] + \\ & + \mathcal{F}_4 [(B^2 k)^\mu (B^2 k)^\nu - (B^2)^{\mu\nu} (k B^2 k)] \} \end{aligned} \quad (33)$$

Here we introduced

$$\begin{aligned} -N_0 \equiv \mathcal{F}_1 &= -\cosh(eaTv) \cos(ebTv) + \sin(ebTv) \sinh(eaTv) \coth(eaT) \cot(ebT) \\ N_3 \equiv \mathcal{F}_2 &= -\sinh(eaTv) \sin(ebTv) + \frac{(1 - \cosh(eaTv) \cosh(eaT)) \cos(ebTv) \cos(ebT) - 1}{\sinh(eaT) \sin(ebT)} \\ N_1 \equiv \mathcal{F}_3 &= \mathcal{F}_1 + 2 \frac{\cos(ebT)}{\sinh^2(eaT)} (\cosh(eaT) - \cosh(eaTv)) \\ N_2 \equiv \mathcal{F}_4 &= \mathcal{F}_1 - 2 \frac{\cosh(eaT)}{\sin^2(ebT)} (\cos(ebT) - \cos(ebTv)). \end{aligned}$$

Rewriting the tensor structure of the coefficients \mathcal{F}_3 and \mathcal{F}_4 we obtain

$$\begin{aligned} \{(33)\} = & -N_0 (g^{\mu\nu} k^2 - k^\mu k^\nu) + N_3 [(Ck)^\mu (Bk)^\nu + (Ck)^\nu (Bk)^\mu] + \\ & + N_1 (Ck)^\mu (Ck)^\nu - N_2 (Bk)^\mu (Bk)^\nu. \end{aligned} \quad (34)$$

By introducing the quantities N_i we have made contact with the work of Urrutia [9]. But while this author treats only parallel E and H fields we allow for any field direction. Urrutias's result is reproduced by our formula (33) together with (34) by putting $C_{\mu\nu} = g_\mu^0 g_\nu^3 - g_\nu^0 g_\mu^3$, $B_{\mu\nu} = g_\mu^2 g_\nu^1 - g_\nu^2 g_\mu^1$. We wanted also mention that Gies [10] has given an alternative but much more elaborate, derivation of $\Pi^{\mu\nu}$. Hence we felt that our representation is sufficiently attractive to present it as another example that demonstrates how path integral methods together with world-line techniques can be put to work.

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